MMP Learning Seminar.

Week 44:

Contents:

- Birational automorphisms.
- DCC of volumes.
- Birationally boundedness.

Birational automorphisms of varieties of general type:

Hacon - McKernan - Xu, 2012.

Theorem 1.1: If n is a positive integer, then there exists a constant $c(n)$ such that the birational automorphism group of a general type variety X of dimension n

has at most $c(n) \cdot \mathrm{vol}(X, K_X)$ elements.

Hurwitz: $|G| \leq 84(g-1)$.

Xiao: S smooth proj of gen type. $|G| \leq 42^2 \, \mathrm{vol}(K_S)$.

Theorem 1.4. (DCC of volumes): Fix $n \in \mathbb{Z}_{>0}$.

$\mathfrak{D}$ the set of global quotient (X, Δ) where X is a proj variety of dimension n.

(1). The set $\{ \mathrm{vol}(X, K_X + \Delta) \mid (X, \Delta) \in \mathfrak{D} \}$ satisfies the DCC.

Further, there are constants, $\delta > 0$ and M s.t if $(X, \Delta) \in \mathfrak{D}$. and $K_X + \Delta$ is big. Then:

(2) $\mathrm{Vol}(X, K_X + \Delta) \geq \delta$ and

(3) $\phi_{M(K_X+\Delta)}$ birational.

Log Birationally Bounded Varieties:

A set of pairs $\mathfrak{D}$ is said to be log birationally bounded if there exists (Z, B) a pair with B reduced, and a projective morphism $Z \longrightarrow T$ where T is of finite type, such that for every $(X, \Delta) \in \mathfrak{D}$. there exists a closed point $t \in T$ and a birational map $f : Z_t \dashrightarrow X$ such that supp B_t containts the support of $Ex(f) + f_*^{-1}\Delta$.

Lemma 2.3.2: $\phi_D : X \dashrightarrow \mathbb{P}^N$ defined by $|D|$. and assume its birational onto its image Z. Then $\mathrm{Vol}(D) \geq \deg Z$. In particular, $\mathrm{Vol}(D) \geq 1$

Proof: Assume ϕ_D is a morphism, Z is non-degenerate of degree ≥ 1. From the inclusion $\phi^*(\mathcal{O}_{\mathbb{P}^N}(1)|_Z \hookrightarrow \mathcal{O}_X(D)$, we conclude $\mathrm{Vol}(D) \geq \mathrm{Vol}(\mathcal{O}_{\mathbb{P}^N}(1)|_Z) = \deg Z \geq 1$. $\square$

Example (small volume):

Define $r_0 = 1$ and $r_{n+1} = r_n(r_n+1)$. Let

$$(X,\Delta) = \left(\mathbb{P}^n, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2 + \dots + \frac{r_{n+1}}{r_{n+1}+1}H_{n+1}\right)$$

$H_0, \dots, H_{n+1}$ are general hyperplanes.

We have that $(X,\Delta) \in \mathfrak{D}$, $\mathrm{vol}(X, K_X+\Delta) = \frac{1}{r_{n+2}^n}$.

Theorem 1.8 (Deformation invariance of plurigenera):

$\pi: X \longrightarrow T$ projective morphism of smooth varieties.

(X,Δ) log canonical and snc over T.

(1). Assume (X,Δ) klt and either $K_X+\Delta$ or Δ is big

$m\Delta$ is integral, then $h^0(X_t, \mathcal{O}_{X_t}(m(K_{X_t}+\Delta_t)))$ is independent of $t \in T$.

(2) $\kappa_\sigma(X_t, K_{X_t}+\Delta_t)$ is independent of $t \in T$.

(3) $\mathrm{Vol}(X_t, K_{X_t}+\Delta_t)$ is independent of $t \in T$.

Theorem 1.9 (DCC of volumes on bir bounded):

Fix a set $I \subseteq [0,1]$ which satisfies the DCC

Let $\mathfrak{D}$ be a set of snc pairs which is birationally bounded,

so that for every $(X,\Delta) \in \mathfrak{D}$, $\operatorname{coeff}(\Delta) \subseteq I$

Then the set of volumes $\{ \operatorname{vol}(X, K_X+\Delta) \mid (X,\Delta) \in \mathfrak{D} \}$

satisfies the DCC

Ideas of the proof (1.4).

Tackle Thm (1.4) using similar ideas to AS.

We will try to find a bir bounded family which the same volumes that appear on (1.4).

$(X,\Delta) \in \mathfrak{D}$ $\quad$ (X',Δ') which is birationally bounded.

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ T \end{array} \quad \text{bounded family.}$$

(1.9) invariance of plurigenera

(X',Δ') are birational to a single variety (Z,B).

$(X_i,\Delta_i),\dots$ $\quad$ $f_i : X_i \longrightarrow Z$.

$$K_{X_i} + \Delta_i = f_i^*(K_Z + \Phi_i) + E_i \qquad \Phi_i = f_*\Delta_i \leq B$$

$$E_i = \underbrace{E_i^+}_{\substack{\text{does not} \\ \text{affect volume}}} - E_i^-$$

Use theory of b-divisors + toroidal blow-ups to prove that all these volumes computation can be performed in a single $Z' \to Z$.

From (1.4) to (1.1).

Y has dimension n

$G = \mathrm{Bir}(Y)$, $\quad Y \overset{G\text{-equiv}}{\dashrightarrow} Y'$, $\quad G = \mathrm{Aut}(Y')$.

Replace Y with a G-equivariant resolution Y'.

Now, we assume $G = \mathrm{Aut}(Y)$ and Y is smooth

$Y \longrightarrow X = Y/G$, $\quad K_X + \Delta$ is big.

$$\mathrm{Vol}(Y, K_Y) = |G| \, \mathrm{vol}(X, K_X + \Delta) \geq |G| \, \delta_n$$

$$|G| \leq \frac{1}{\delta_n} \mathrm{Vol}(Y, K_Y).$$

$\underset{\displaystyle c.}{\overset{!!}{}}$

Potentially Birational:

X normal projective, D big $\mathbb{Q}$-Cartier, $x, y \in X$ very general

assume we can find $0 \leq \Delta \sim_{\mathbb{Q}} (1-\varepsilon)D$ for some $0 < \varepsilon < 1$.

where (X, Δ) is not klt at y & (X, Δ) is lc at x

and $\{x\}$ is an isolated log canonical center. Then, we say that

D is potentially birational.

Lemma 2.3.4: X normal g.p variety of dim n. D big on X

(1) D is potentially birational $\Longrightarrow$ $\phi_{K_X + \lceil D \rceil}$ is birational.

(2) ϕ_D is birational $\Longrightarrow$ $(2n+1)\lfloor D \rfloor$ is potentially bir

(3) ϕ_D is birational $\Longrightarrow$ $\phi_{K_X + (2n+1)D}$ is bir

In particular, $K_X + (2n+1)D$ is big.

Theorem 3.2.5: (X, Δ) klt, $(X, \Delta + \Delta_0)$ lc around x &

non-klt at y, V non-klt center which contains x.

H ample with $\mathrm{Vol}(V, H|_V) > 2k^k$, where $k = \dim V$.

There exists, $H \sim_{\mathbb{Q}} \Delta_1 \geq 0$, $0 \leq a_1 \leq 1$, so that

$(X, \Delta + a_0 \Delta_0 + a_1 \Delta_1)$ is around x and non-klt at y

and a non-klt center that contains x has dim $< k$.

Theorem 2.3.6: (X, Δ) klt pair, where X has dim n.

H ample, $\gamma_0 \geq 1$ such that $\mathrm{vol}(X, \gamma_0 H) > n^n$.

$\varepsilon > 0$ with the following property:

$x \in X$ very general, for every $0 \leq \Delta_0 \sim_{\mathbb{Q}} \lambda H$ s.t. $(X, \Delta + \Delta_0)$ is lc at x and V is a minimal lc center containing x. Then $\mathrm{vol}(V, \lambda H|_V) > \varepsilon^k$ where k is the dimension of V and $\lambda \geq 1$.

Then mH is potentially birational, where $m = 2\gamma_0 (1+\gamma)^{n-1}$

$$\gamma = 2n / \varepsilon.$$

Idea: Descending induction on k.

Claim: There exists $\Delta_0 \sim_{\mathbb{Q}} \lambda H$ with $1 \leq \lambda < 2\gamma_0 (1+\gamma)^{n-1-k}$ with $(X, \Delta + \Delta_0)$ lc at x non-klt at y and a non-klt center V of dim $\leq k$ contains x.

Properties of birationally bounded families:

Lemma 2.4.2: $\mathfrak{X}, \mathfrak{Y}$ are classes of varieties (or pairs) of dimension n.

(1) $\mathfrak{X}$ bir bounded, $\forall\, Y \in \mathfrak{Y}$, Y is birational to $X \in \mathfrak{X}$. Then $\mathfrak{Y}$ is bir bounded.

(2) $\forall\, X \in \mathfrak{X}$, there exists D Weil with ϕ_D birational and $\mathrm{vol}(D) \le V$, then $\mathfrak{X}$ is bir bounded.

(3) $\mathfrak{X}$ is log bir bounded, $\forall\, (Y, \Delta_Y) \in \mathfrak{Y}$, there exists $(X, \Delta) \in \mathfrak{X}$ with $f : X \dashrightarrow Y$ birational map s.t. Δ contains $f_*^{-1}\Delta_Y$ and $\mathrm{Ex}(f)$. Then $\mathfrak{Y}$ is log birationally bounded.

(4). $\mathfrak{X}$ is log bir bounded $\quad \{X \mid (X, \Delta) \in \mathfrak{X}\}$ is bir bounded.

(5) $(X, \Delta) \in \mathfrak{X}$, there exists a Weil D, with $\phi_D : X \dashrightarrow \mathbb{P}^N$ birational onto its image s.t.

$D = K_X + m(K_X + \Delta)$

$\boxed{\mathrm{Vol}(D) \le V_1}$ if $G = \mathrm{Ex}(\phi_D^{-1})_{red} + \phi_{D*}\Delta_{red}$ then $G \cdot H^{n-1} \le V_2$. where H is the ample defined by D.

Then $\mathfrak{X}$ is birationally log bounded.

Birationally bounded pairs:

Theorem 3.1: Fix $n, A, \delta > 0$. The set of log pair (X, Δ) satisfying the following conditions:

(1) X is projective of dim n,

(2) (X, Δ) is lc,

(3) $\mathrm{Coeff}\, \Delta \geq \delta$,

(4) there exists $m \in \mathbb{Z}_{>0}$ with $\mathrm{vol}(X, m(K_X + \Delta)) \leq A$ and

(5) $\phi_{K_X + m(K_X+\Delta)}$ is birational.

Is log birationally bounded.

Lemma 3.2: X normal proj of dim n.
M bpf Cartier and ϕ_M is birational. Set $H = 2(n+1)M$.
If D is a sum of distinct prime divisors, then

$$D \cdot H^{n-1} \leq 2^n \operatorname{vol}(X, K_X + D + H).$$

Proof: (X, D) log smooth, comp of D disjoint

No component of D is contained in the exceptional of ϕ_M

$M \sim_{\mathbb{Q}} A + \underline{\underline{B}}$, $\quad K_X + \underline{D} + \delta \underline{B}$ is dlt for $\delta \ll 1$.

$$H^i(K_X + E + pM) = 0, \quad p \geq 0, \ i > 0 \quad 0 \leq E \leq D.$$

(2) of (2.3.4). imply that $K_X + D + H =: A_1$ is big, so it has an ample model

$$Q(m) = h^0(X, \mathcal{O}_X(2m A_1)).$$

Set $A_m = K_X + D + mH$, so $H^i(D, \mathcal{O}_D(A_m)) = 0$.

$P(m) = h^0(D, \mathcal{O}_D(A_m))$ is a polynomial on m.

Leading terms:

	Q	P
	$\dfrac{2^n \operatorname{vol}(K_X + D + H)}{n!}$	$\dfrac{D \cdot H^{n-1}}{(n-1)!}$

$t \in H^0(2mA_1 - A_m)$ does not vanish on components of D.

We have a commutative diagram:

$$H^0(-) \twoheadrightarrow H^0(-)$$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(A_m - D) \arrow[r] \arrow[d] & \mathcal{O}_X(A_m) \arrow[r] \arrow[d] & \mathcal{O}_D(A_m) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X(2mA_1 - D) \arrow[r] & \mathcal{O}_X(2mA_1) \arrow[r] & \mathcal{O}_D(2mA_1) \arrow[r] & 0 \\
 & & & s \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[ul, bend left, "\text{lift}"'] &
\end{tikzcd}$$

is in the image of the vertical map

$$P(m) \leq h^0(X, \mathcal{O}_X(2mA_1)) - h^0(X, \mathcal{O}_X(2mA_1 - D)).$$

$$P(m) \leq \underbrace{Q(m) - Q(m-1)}_{= Q'(m).}$$

$\square$

Theorem 3.1: Fix $n, A, \delta > 0$. The set of log pair (X, Δ) satisfying the following conditions:

(1) X is projective of dim n,

(2) (X, Δ) is lc,

(3) Coeff $\Delta \geq \delta$,

(4) there exists $m \in \mathbb{Z}_{>0}$ with $\mathrm{vol}(X, m(K_X + \Delta)) \leq A$ and

(5) $\phi_{K_X + m(K_X+\Delta)}$ is birational.

Is log birationally bounded.

Proof: $\phi = \phi_{K_X + m(K_X+\Delta)}$ is a morphism $X \xrightarrow{\phi} Z$.

$|K_X + m(K_X+\Delta)| = |M| + E$, $\quad M = \phi^* H$.

$$\boxed{\mathrm{Vol}(K_X + m(K_X+\Delta)) \leq \mathrm{vol}((m+1)(K_X+\Delta)) \leq 2^n A.}$$

$\boxed{G = \phi_* \Delta_{red}}$ $\quad B \in ||K_X + (m(K_X+\Delta))||$.

$$\alpha = \max\left(\frac{1}{\delta}, 2(2n+1)\right).$$

$D_0 =$ reduced sum of comp of Δ and B which are not contracted by ϕ.

$$D_0 \leq \alpha(B + \Delta)$$

$\alpha(m+1)(K_X+\Delta) - \alpha(B+\Delta) \sim_{\mathbb{Q}} C \geq 0.$

$\boxed{\text{Compute } H^{n-1} G}$

$$
\begin{aligned}
G \cdot H^{n-1} &\leq \overbrace{D_0}^{D} \cdot \overbrace{(2(2n+1)M)}^{H}{}^{n-1} \\
&\leq 2^n \operatorname{vol}(X, K_X + D_0 + 2(2n+1)M) \\
&\leq 2^n \operatorname{vol}(X, (1+2\alpha(m+1))(K_X+\Delta)) \\
&\leq 2^n (1+2\alpha(m+1))^n \operatorname{vol}(K_X+\Delta) \\
&\leq 2^{3n} \alpha^n \operatorname{vol}((m+1)(K_X+\Delta)) \\
&\boxed{\leq 2^{4n} \alpha^n A.}
\end{aligned}
$$

$\Rightarrow$ only depends on A, δ and n

$\square$.